

T Curtright & C Zachos (ANL), [arXiv:1010.5174] PhysRevD83 (2011) 065019; & [arXiv:0909.2424] JPhysA42 (2009) 485208; [1002.0104] JPhysA43 (2010) 445101

THE GLOBAL STRUCTURE OF THE RENORMALIZATION GROUP

The Gell-Mann–Low finite renormalization group equation (QED, 1954),

$$\Psi(g(t)) = \lambda^t \Psi(g),$$

(where t is the logarithm of the distance/energy reference point, λ sets its scale, $g \equiv g(0)$, and Ψ is the RG “scaling function”, which “rectifies” group flow to linear flow) is usually solved by integrating a perturbative approximant to its algebra, the β -function,

$$\frac{dg}{dt} = \beta(g) \equiv (\ln \lambda) \Psi(g) / \Psi'(g),$$

in g , to obtain the full **RG trajectory**, for $\lambda \neq 1$,

$$g(t) = \Psi^{-1}(\lambda^t \Psi(g)).$$

‡ But there is a **different way** to calculate and analyze the RG trajectory, based on the theory of this functional conjugacy equation, which was actually introduced by E Schröder (1870). For $f(g) = g(1)$,

$$\lambda\Psi(g) = \Psi(f(g)).$$

✓ In this conjugacy form, the global self-similar functional structure of the RG trajectory is more apparent, and illuminates the **interplay between continuous and discrete rescaling** (step-scaling $f(g)$ in lattice gauge theory or chaotic maps), often inaccessible to conventional local relations.

↷ **an analytic interpolate** between $g = g(0)$ and $f(g) = g(1)$, just from boundary (“holographic”, discrete) data, **without the benefit of a local propagation relation**: $g(t) = \Psi^{-1}(\lambda^t\Psi(g))$.

► Essentially, taking arbitrary functional roots of an arbitrary function f ; e.g., $\text{rin}(x)$, the functional square root ($t = 1/2$) of

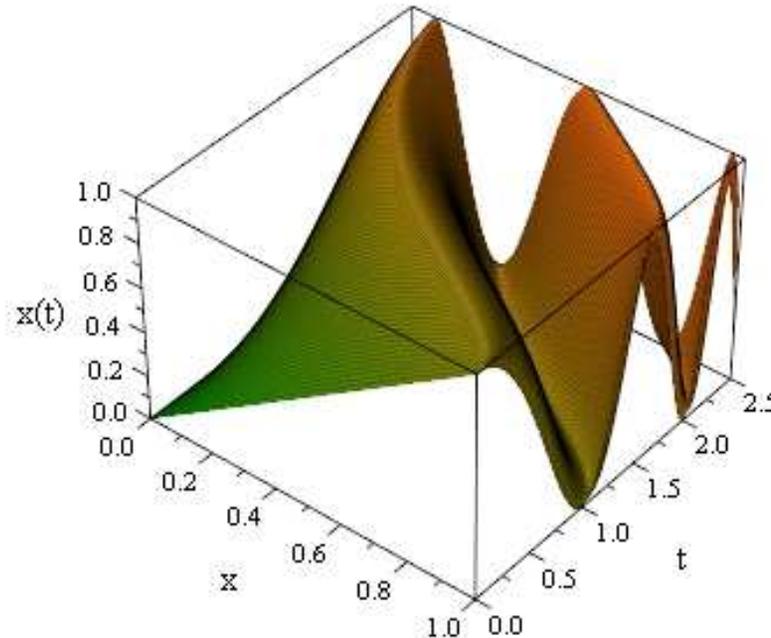
$$\text{rin}(\text{rin}(x)) = \sin(x).$$

↗ E.g., can do this for the logistic map in the chaotic regime (imaginary magnetic field Ising model), $r = 4$,

$$g(1) = 4g(1 - g) \equiv f_1(g),$$

$g(t + 1) = f_1(g(t)) = f_1(f_t(g)) = f_{t+1}(g)$, an associative and commutative group composition, \circlearrowright . ↗

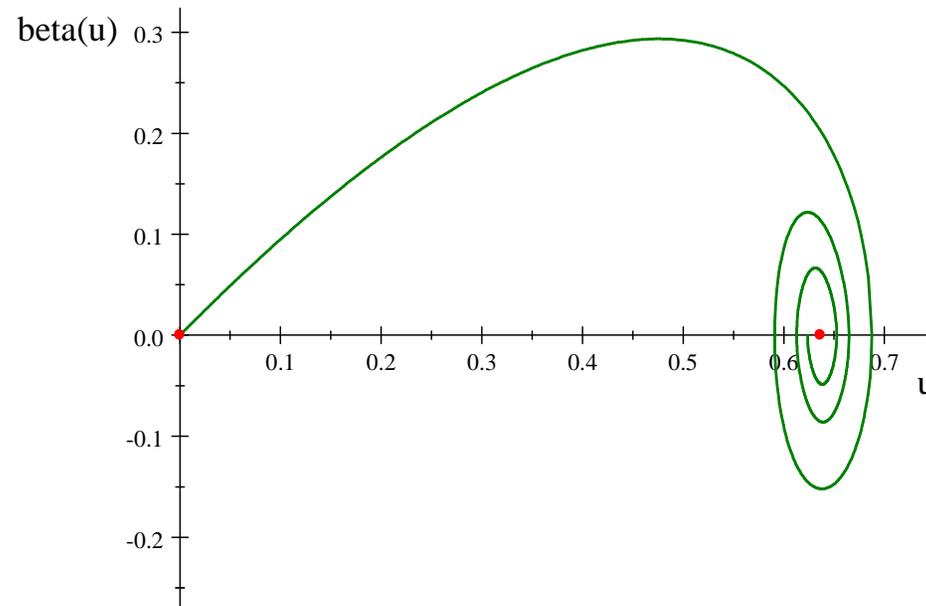
$$f_t(g) = g(t) = \sin^2(2^t \arcsin(\sqrt{g})),$$



⊗ One can now infer the β -function (velocity) and hence manufacture the underlying Hamiltonian dynamical system which yields this RG flow by

conserving energy. ○ Specific dynamics as an emergent phenomenon: Analogy to inverse scattering: initial and final profiles yield a potential.

Such trajectories can be **multivalued**. E.g. for nonchaotic $r = 11/4$,



⊙ **The method:** Consider analytic $f_t(g)$ around a **fixed point** of $f(g)$. Without loss of generality, take the fixed point to be $g = 0$: $f(0) = 0$, $\rightsquigarrow \Psi(0) = 0$, and if $\Psi'(0) \neq 0, \infty$, then $\lambda = f'(0)$. Solve for $\Psi(g)$ in terms of $f(g)$, if needs be by recursion of the respective series coefficients. Finally, invert to obtain Ψ^{-1} , and set $\lambda \rightarrow 1$ if the problem requires it—the answer may be convergent even if Ψ diverged for $\lambda = 1$. The group orbit found is thus **analytic around the fixed point** $g = 0$.

★ The β -function is then an **emergent feature**,

$$\beta(g) = \ln \lambda / (\ln \Psi(g))'$$

(and, e.g., could be obtained from a lattice step-scaling function.)

⊃ **The Meaning** of the scaling variable Ψ : It's but the conjugacy **variable transformation** $w = \Psi(g)$ which **trivializes the action of $f(g)$ to a mere scaling** $w \mapsto \lambda w$, ("rectification")

\leadsto trivial to iterate $\circlearrowleft \forall t$:

$$\begin{array}{ccc} g & \xrightarrow{f} & f(g) \\ \Psi(x) \downarrow & & \downarrow \Psi(f(g)) \\ w & \xrightarrow{\lambda} & \lambda w \end{array}$$

▲ The composite map is then $g \mapsto \Psi(f(g)) = \lambda \Psi(g)$. \curvearrowright
 $g(t) = \Psi^{-1}(\lambda^t \Psi(g))$.

⊛ Moreover, **nonlocal relations follow**, such as the Julia equation,

$$\beta(g(1)) = \frac{\partial f}{\partial g} \beta(g),$$

\curvearrowright Extrema of $f(g)$ imply zeros of $\beta(g(1))$, before obtaining Ψ .

✓ Periodic Ψ^{-1} s yield **limit cycles** (even) for real coupling, cf. the “Russian Doll” superconductivity model of LeClair, Román, and Sierra (2004), $\Psi^{-1} = \tan \log \quad \rightsquigarrow$ periodicity of the physics in t , the logarithm of the scale μ :

$$g(t) = \tan(t \log \lambda + \arctan g) .$$

\rightsquigarrow **The physics repeats itself cyclically** in self-similar modules.

► Constancy of Ψ^{-1} , instead, yields **fixed points**. However, solutions thus found may have novel, exotic features, including multiple branches: zeroes of β do *not* necessarily signal fixed points of the flow, but instead (if the acceleration or higher derivatives do not vanish), may only indicate **turning points** of the RG trajectories, explored by the novel functional conjugacy methods.

→ Revealing intriguing **multivalued** behaviors including chaotic (spin-glass) RG flows.