T Curtright & C Zachos (ANL), [arXiv:1010.5174] PhysRevD83 (2011) 065019; & [arXiv:0909.2424] JPhysA42 (2009) 485208; [1002.0104] JPhysA43 (2010) 445101; [arXiv:1105.3664]

THE GLOBAL STRUCTURE OF THE RENORMALIZATION GROUP

Essential progress in particle physics may hinge on new quantum field theories and the understanding of their **renormalization group flows**.

► A general program of studying these systematically **in the large** has been initiated and produced several results and insights.

✓ Potential lattice applications.

The Gell-Mann–Low finite renormalization group equation (QED, 1954),

$$\boxed{\Psi\left(g\left(t\right)\right) = \lambda^{t-t_0}\ \Psi\left(g\right)}\ ,$$

(where $t = \ln \mu$ of the distance/energy reference point; λ sets its scale; $g \equiv g(t_0)$, @ arbitrary $t_0 = 0$; Ψ is the RG "scaling function", which "rectifies" group flow to linear flow) is usually solved by integrating a perturbative approximant to its algebra, the β-function,

$$\boxed{\frac{dg}{dt} = \beta\left(g\right) \equiv \left(\ln\lambda\right)\Psi\left(g\right)/\Psi'\left(g\right)}\ ,$$

in g, to obtain the **full RG trajectory**, for $\lambda \neq 1$,

$$\boxed{g(t) = \Psi^{-1}(\lambda^t\Psi(g))}\ .$$

⫛ But there is a **different way to calculate** and analyze the RG trajectory, based on the theory of this functional conjugacy equation, which was actually introduced by E Schröder (1870). For a discrete leap $f(g) = g(1)$,

$$\boxed{\Psi(f(g)) = \lambda \Psi(g)} \ .$$

✓ In this conjugacy form, the global self-similar functional structure of the RG trajectory is more apparent, and illuminates the **interplay between continuous and discrete rescaling** (step-scaling $f(g)$ in lattice gauge theory or chaotic maps), often inaccessible to conventional local relations.

$\rightsquigarrow$ **an analytic interpolate** between $g = g(0)$ and $f(g) = g(1)$, just from boundary ("holographic", discrete) data, **without the benefit of a local propagation relation**: $g(t) = \Psi^{-1}(\lambda^t \Psi(g))$.

▶ Essentially, taking **arbitrary functional roots** of an arbitrary function f; e.g., rin(x), the functional square root $(t = 1/2)$ of

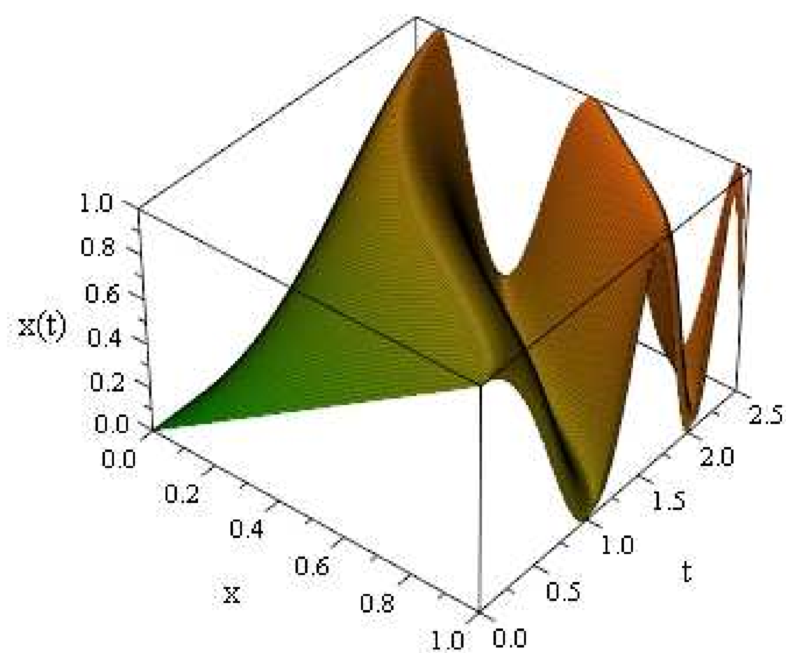
$$\boxed{\text{rin}(\text{rin}(x)) = \sin(x)}.$$

$\looparrowright$ E.g., can do this for the logistic map in the chaotic regime (imaginary magnetic field Ising model), $r = 4$,

$$\boxed{g(1) = 4g(1-g) \equiv f_1(g)},$$

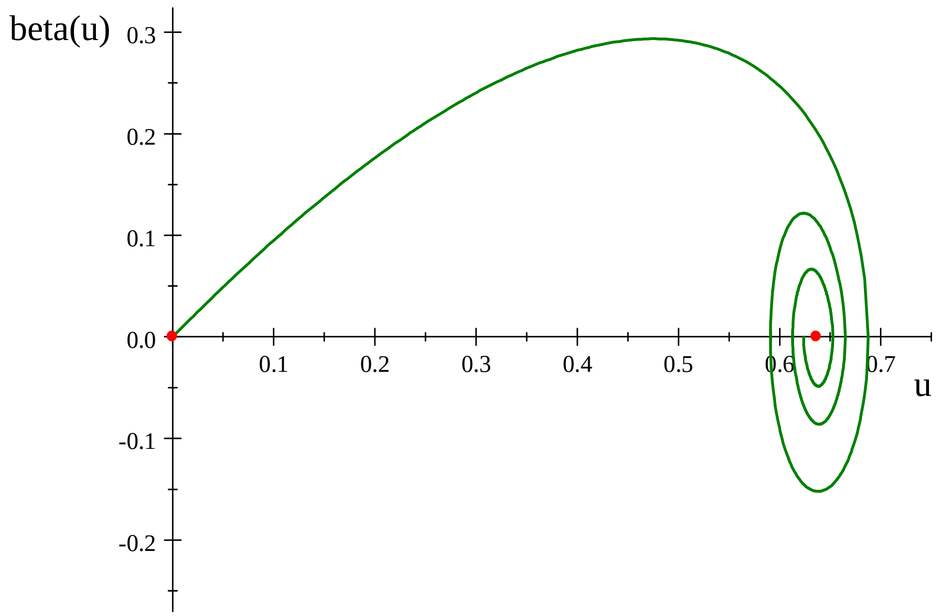
$g(t+1) = f_1(g(t)) = f_1(f_t(g)) = f_{t+1}(g)$, an associative and commutative group composition, $\circlearrowleft$. $\looparrowright$

$$\boxed{g(t) = f_t(g) = \sin^2(2^t \ \arcsin(\sqrt{g}))},$$



$\circledast$ One can now **infer** the β-function (velocity) and hence manufacture an underlying Hamiltonian dynamical system which yields this RG flow by conserving energy. ⥁ Specific dynamics as an ***emergent phenomenon***: Analogy to inverse scattering: initial and final profiles yield a potential.

Such trajectories can be ***multivalued***. E.g. for nonchaotic $r = 11/4$,



⊚ ***The method***: Consider analytic $f_t(g)$ around a ***fixed point*** of $f(g)$. Without loss of generality, take the fixed point to be $g = 0$: $f(0) = 0$, $\rightsquigarrow$ $\Psi(0) = 0$, and if $\Psi'(0) \neq 0, \infty$, then $\lambda = f'(0)$. Solve for $\Psi(g)$ in terms of $f(g)$, if needs be by recursion of the respective series coefficients. Finally, invert to obtain Ψ^{-1}, and set $\lambda \to 1$ if the problem requires it—the answer may be convergent even if Ψ diverged for $\lambda = 1$. The group orbit found is thus ***analytic around the fixed point*** $g = 0$.

★ The β-function is then an **emergent feature**,

$$\boxed{\beta(g) = \ln\lambda / (\ln\Psi(g))'}\ ,$$

(and, e.g., could be obtained from a lattice step-scaling function.)

⋔ **The Meaning** of the scaling variable Ψ: It's but the conjugacy **variable transformation** $w = \Psi(g)$ which **trivializes the action of** $f(g)$ **to a mere scaling** $w \longmapsto \lambda w$, ("rectification")
⇝ trivial to iterate ↺ $\forall t$:

$$\begin{array}{ccc} g & \xrightarrow{f} & f(g) \\ \Psi(x) \downarrow & & \downarrow \Psi(f(g)) \\ w & \xrightarrow{\lambda} & \lambda w \end{array}$$

▲ The composite map is then $g \longmapsto \Psi(f(g)) = \lambda\Psi(g)$.
↷ $g(t) = \Psi^{-1}(\lambda^t \Psi(g))$.

⊛ Moreover, **nonlocal relations follow**, such as the Julia equation,

$$\boxed{\beta(g(1)) = \frac{\partial f}{\partial g}\ \beta(g)},$$

↷ Extrema of $f(g)$ imply zeros of $\beta(g(1))$, before obtaining Ψ.

✓ Periodic Ψ^{-1}s yield **limit cycles** even for a real coupling, cf. the "Russian Doll" superconductivity model of LeClair, Román, and Sierra (2004), $\Psi^{-1} = \tan\log$ $\rightsquigarrow$ periodicity of the physics in t, the logarithm of the scale μ:

$$\boxed{g(t) = \tan(t \ \log\lambda + \arctan g)} \ .$$

$\rightsquigarrow$ **The physics repeats itself cyclically** in self-similar modules.

▶ Constancy of Ψ^{-1}, instead, yields **fixed points**. However, solutions thus found may have novel, exotic features, including multiple branches: zeroes of β do *not* necessarily signal fixed points of the flow, but instead (if the acceleration or higher derivatives do not vanish), may only indicate **turning points** of the RG trajectories, explored by the novel functional conjugacy methods.

$\twoheadrightarrow$ Revealed intriguing **multivalued** behaviors including chaotic (spin-glass) RG flows.